Revisiting the Bid-ask Spread Using Competitive Search

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Abstract

In this paper, we set up a competitive search model to re-interpret the existence of the market equilibrium bid-ask spread in a stylized security market, in which market dealers are in charge of posting an instantaneous bid price, investors choose whether to sell their share or not at this price, and the fluctuation of market sentiment is mimicked by an arrival rate of arbitrage opportunities with a Poisson process. Different from the asymmetric information based explanation originated from two types of investors, our search based model emphasizes that since the market dealer provides necessary liquidity to the security market via playing such an intermediary role between actual buyers and sellers, the bid-ask spread charged thereafter should largely be justified as the compensation for the market dealer’s endeavor in this process. Our model provides a closed-form bid-ask spread formula which has a capacity to reproduce many empirical observations with respect to the effects of the market dealer’s maintenance cost, the discount rate prevalent in the market, the overall market uncertainty, and the dividends payoff on the magnitude of the bid-ask spread. Our model further indicates that the absolute bid-ask spread is positively related to the stock price level while the percentage bid-ask spread is negatively related to the stock price level, which solves the puzzle on the impact of stock splits on stock liquidity without the assumption of asymmetric information.

Key words: competitive search, bid-ask spread, market microstructure, market dealer, stock liquidity, stock splits

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As traditional capital asset pricing models (CAPM) only consider the market systematic risk that is compensated by the equity risk premium (ERP), they cannot explain a variety of pricing puzzles and anomalies associated with the cross-sectional expected stock returns. Many researchers found that liquidity premium is one of the important explanatory factors but ignored in CAPM(Pastor (2003)). While we all can feel that financial assets differ in their liquidity such as some of stocks are much easier to trade and the others are not if without a longer waiting time or a larger price impact, the exact meaning of liquidity is rather elusive. The origin of liquidity difference of different assets is not what we are going to investigate here. Insofar as the liquidity of a stock is typically measured by its bid-ask spread in literature(Amihud (1986)), rather, in this paper, we will focus on one more easily tackled but correlated question: why does there exist a bid-ask spread for a traded asset and how to quantify it theoretically?

Although there are many papers focusing on the market microstructure trying to interpret why there exists a bid-ask spread in a security market, including one popular model of bid-ask spread proposed by Kyle(Kyle (1985)), which is mainly based on the asymmetric information, showing that dealers have to widen the bid-ask spread in order to trade against informed investors, our paper here attempts to treat the bid-ask spread from a different viewpoint, i.e. search-based angle, which doesn’t mean that the issue of asymmetric information between dealers and investors is not important. Unfortunately we, however, avoid the trace of asymmetric information deliberately in our model in order to emphasize more insightful and more fundamental search and matching characteristics of the market equilibrium of the bid-ask spread phenomenon.
Introduced here as the background information, in macroeconomics, search theory is widely used to explore the matching behavior between workers and firms. The typical framework is in this way: there are two types of rational agents, workers and firms; they meet with each other depending on the current market tightness; the pure meeting of each other doesn’t necessarily lead to a permanent job contract since the worker would expect that he or she might come across a better job offer if he or she just waits a little longer, in the meantime the firm has the same thought to sign a higher productive worker as long as it is more patient(Diamond (1984 )).

With regard to the search mechanism between workers and firms, there are two key modeling issues which can significantly influence the final jointly searching results: (1) how workers and firms meet with each other and (2) how they decide the wage (i.e. how they split the total profit from a successful hiring). Minor differences in institution arrangements may lead to diverse predictions. Two possible arrangements have been thoroughly analyzed in search literature: one is called “random search” in which workers and firms just randomly meet with each other and the final wage for workers is determined via the generalized Nash bargaining scheme once they meet; the other is called “competitive search” in which workers’ job search activities are not random but directed to specific firms either because those firms post and advertise their offered wages or because those firms are natural focal points due to history or established reputation (Rogerson, Shimer and Wright 2005).

To the best of our knowledge, We are the first to resort to competitive search, also known as directed search with price posting, to investigate the interaction between market dealers and investors in a stylized security market in order to re-interpret the existence of the bid-ask spread
at the market equilibrium. The common image for our competitive search is: assuming that only one type of asset is being traded between market dealers and investors, each market dealer first posts its own bid price and will commit to it without collusion with any other market dealer; since there are many market dealers, there will be many possible sub-markets and each sub-market is distinguished by its own posted bid price; in doing so, the market dealers anticipate that investors will enter until investors are indifferent across all sub-markets. The market equilibrium of our competitive search is: there will, however, be only one sub-market with one bid price, such that no market dealer has any incentive to open a new sub-market by posting a bid price different from the uniform one, meanwhile no investors want to enter that newly opened sub-market.

The center theme delivered by search theory is that it always takes time for one agent to locate another one if the success of a transaction requires the collaboration of both agents. Thus, the difficulty of identifying the counter-party of the transaction (either buying or selling a security) in a security market justifies a role of a market intermediary who is introduced to facilitate the easiness of buying and selling through shortening the time wasted in matching and dampening the price impact of a larger order.

The above represents the concept of market search friction that exists in any security market, but more saliently exists in over the counter (OTC) market (Vayanos (2008)) and (Lagos (2009)), government or corporate bonds market, foreign exchange market and NASDAQ stock market than in NYSE market where the security transaction is more centralized.

It is admitted that with the rapid progress of advanced IT technology and more centralized
market transactions, searching for the counter-party has become more efficient. While the market search friction plays a less important role for the operations of security markets than previously, we, however, can still appreciate the merits of search based market microstructure modeling from two aspects:

Firstly, centralized markets historically all evolve from initial de-centralized markets. When considering the compensation for a role of a market dealer playing in a centralized market as the form of the bid-ask spread, we need to compare two states, one is the real centralized market, and the other is the imaginary de-centralized market, even if the de-centralized market is not the current status of the market transaction. To be more specific, only if we explore how difficult it is to meet the other side of the transaction in the imaginary de-centralized market, can we then evaluate accurately how well the liquidity that the real centralized market provides to investors, and justify the amount of the bid-ask spread required by a market dealer in the current centralized market. Hence, the market search friction modeled in this manuscript should be embedded implicitly in the bid-ask spread for any centralized market.

Secondly, our competitive search based model well characterizes the matching process between market dealers and investors. As we have mentioned before, our competitive search based model is distinguished from random search based models by how the search process proceeds. While random search based models assume 1) the random matching between the types of agents such as workers and firms, 2) the determination of price by bargaining once they meet, which are not fit into our market microstructure environment, our competitive search based model allows market dealers to post a widely known bid price in public ex ante in order to direct
or attract the arrival of investors. Furthermore, our symmetric equilibrium lets all market dealers play the same strategy at the market equilibrium, i.e. post the same bid price in the entire market. In this way, our search based model well mimics the operation of NASDAQ stock market.

The existence of market dealers for any security is so indispensible and so natural that it is hard to image what would happen if there were not such a role there. As we never take a pause to challenge why the New York Stock Exchange charges a service fee due to the secondary market liquidity provided to the entire society by it, so in the same logic, any reasonable explanation of why the bid-ask spread is charged by market dealers should be traced back to the service offered and the role played by market dealers, and the function of asymmetric information matters only to modulate the above fundamentals, which is the basic belief of the authors and the motivation of this paper as well.

Different from traditional bid-ask spread theories which pay much attention to the inventory-holding cost and the asymmetric information cost associated with two types of investors in the market, the key contribution of our paper is that we emphasize that the bid-ask spread can mainly be justified as the compensation for market dealers’ endeavor in providing necessary liquidity to the security market via playing an intermediary role between actual buyers and sellers. In another word, our model stresses the importance of searching and matching cost. Our model further shows that “peer pressure” resulted from competition from other market dealers can downplay their role as a market intermediary while they are always willing to charge investors with the highest possible bid-ask spread.

Our model provides a closed-form bid-ask spread formula based on competitive search in a
stylistic security market, the simulation results of which are well consistent with the real security markets, thus proving the validity of our model.

Specifically, our model shows that the maintenance cost has two opposite effects on the bid-ask spread. Its direct effect on the bid-ask spread is positive, which is in agreement with the predication of inventory holding cost theory of bid-ask spread. Our model also indicates that the indirect effect of the maintenance cost is negative since higher maintenance cost could reduce the total number of market dealers, which also represents a more competitive structure for market dealers, leading to a less amount of bid-ask spread. Overall, the positive direct effect of the maintenance cost on the bid-ask spread dominates the negative indirect effect according to our simulation results.

If we treat the dividends payoff as the “negative” maintenance cost, it is natural for our competitive search based model to acquire the positive effect of the dividends payoff on the bid-ask spread. While asymmetric information based models reach the same conclusion, the underlying mechanism is totally different as those information based models presume that there exists a positive relation between the level of information asymmetry and the magnitude of the bid-ask spread. Thus larger amount of dividends payoff signals the market the less information asymmetry, causing the bid-ask spread to shrink.

Our model also explicitly studies the influences of the discount rate and the instantaneous market opportunity on the bid-ask spread. Our model indicates that both the discount rate and the instantaneous market opportunity have a positive effect on the bid-ask spread. The above results are obvious if we consider the discount rate as the opportunity cost of holding stocks without
earning the interest rate of a market dealer’s own fund meanwhile if we treat the instantaneous market opportunity as the measure of the overall market uncertainty.

More importantly, our model studies the impact of the price level on the bid-ask spread, which is closely related to a very important and prevalent financial market phenomenon both in the U.S. and worldwide-\textit{stock splits}. People show great interest in the relationship between stock splits and stock liquidity. Contrary to the confusing explanations and observations originated from varieties of asymmetric information based models, our competitive search model clearly shows that the (absolute) bid-ask spread is positively related to the stock price level, but the percentage bid-ask spread is negatively related the stock price level.

In addition, better than conventional market microstructure models, our model has a capacity to determine the total number of market dealers at the market equilibrium since we don’t let the total number of market dealers fixed beforehand when modeling the dynamic interaction between market dealers and investors in a security market. This number can be pinned down by the system via a free entry and exit condition for market dealers, which is also the reason why our model is named as the “competitive” search model.

The rest of this paper proceeds as follows: section 1 sets up our competitive search model; section 2 solves the model and derives the main results; section 3 discusses the empirical implications of the model; section 4 calibrates the key parameters of the model and illustrates the effects of many factors on the bid-ask spread, and section 5 concludes and points the future directions. Symbols and notations are summarized in Appendix A. Proofs of propositions are provided in Appendix B.
1. Competitive search model

1.1 Assumptions

In a simplified world of a security market with only one type of asset or stock to be traded, we have many market dealers (denoted by f) who publicly post an instantaneous bid price (W), at which many potential investors (denoted by w) would choose whether to sell their share or not at this price. We assume that each investor initially has only one share of the stock inhered from endowment; each market dealer can only serve one transaction at a time and each transaction consists of only one share of the stock.

Furthermore, assume that time is continuous and goes from zero to infinity, both agents (f and w) are risk neutral with the (risk-free) discount rate of r. In addition, even though heterogeneity of agents is more realistic, both agents are assumed to be homogenous in this model respectively.

Let the number of investors be normalized to 1. During each time period, (1-u) of them sell their shares to market dealers and the left of them (u) don’t sell their shares to market dealers. Thus, the number of market dealers who are occupied by stock transactions has to be (1-u). If we assume that the number of market dealers who are idle is assumed to be v, then the total number of market dealers for this stock will be (1-u) +v=1-u+v. We define the market tightness as $\theta=\frac{v}{u}$, one key system variable which characterizes the tightness of the market condition, namely, the higher this ratio is, the more the number of idle market dealers and the fewer the number of investors who still have shares of the stock to be sold out, hence it is easier for any investor to sell his or her share if he or she is willing to.
Another note to be emphasized is that since the basic structure of selling and buying is similar in nature, the following model will only be concentrated on one half of the market activities, i.e. the selling part of the market. Namely, we only consider the relation between market dealers and investors in which market dealers post an instantaneous bid price and investors decide whether to sell the share or not. The modeling of the buying part of the market in which market dealers post an instantaneous ask price and investors decide whether to buy the share or not will not become too insurmountable if the selling part of the market is clearly understood. Therefore, from this viewpoint, our model can be classified into a class of partial equilibrium models.

1.2 General picture

In sum, the general picture is: there are two types of agents continuously interacting with one another in the selling part of the market. They need to match with each other to realize their respective optimal profits.

Investors, whose final aim is to sell one share of the stock they obtain from the initial endowment. If an investor holds the share, each period he or she will extract \( b \) units of utility (or money) forever. We can think of one share of the stock as one “Lucas tree” which can produce \( b \) units of dividends during each time period and \( b \) has the same unit as the bid price \( W \). However, with an arrival rate of \( m \) that is an increasing and concave function of the market tightness \( \theta \), the investor may change his or her mind and sell the share to a market dealer at any bid price \( W \) currently posted in the market. To be stressed here, in the strictest term, our investors’ search behavior is not random but directed by the bid price \( W \) publicly known, which is a key difference
between our model and other search based models in Finance pioneered by Duffie (Duffie (2005)) and Weill (Weill (2008)).

Market dealers play an intermediary role by posting a bid price $W$ signaling that they are always available to buy investors’ shares at $W$. Market dealers need to compete with each other for providing the “liquidity service” to all investors, which implicitly determines the number of market dealers this security market can finally support at the market equilibrium. A market dealer incurs a maintenance cost of $a$ by posting a bid price $W$ in the market. When the transaction is settled, the market dealer will pay the bid price $W$ during each time period to the investor in exchange for the share which can be sold later at the price $P$ by the market dealer then. (Since we only consider the selling part of the market, from now on, the price $P$ that the market dealer can realize in the other side of the market will be treated as a parameter in our model while the bid price $W$ is still a choice variable.) The price difference between $P$ and $W$, titled with the bid-ask spread, compensates the market dealer for working as a counter party in any stock selling transaction. In the language of economics, $(P - W)$ can also be considered as the normal profit of the market dealer.

With an arrival rate of $\lambda$ that follows a standard Poisson process, the market dealer who holds one share from the previous transaction cancels this transaction, i.e. stop paying $W$ afterwards. Different from traditional market microstructure models where the overall market sentiment is represented by the arbitrage transaction opportunities that investors are facing, our model, however, stresses the market sentiment embedded in the market dealers’ transaction decisions. Through this delicate mechanism design, our model is endowed with the power to mimic the
instantaneous arbitrage opportunities for the market dealer due to the fluctuation of the entire market sentiment.

To be clarified here, the price \( P \) at which the market dealer can sell out the share, the bid price \( W \) posited by the market dealer, the maintenance cost \( a \) occurring to the market dealer when posting a bid price in the market, and the dividends \( b \) produced by one share of the stock are not "stock variables" but "flow variables" in response to the time factor in our model. Both market dealers and investors are constrained by the basic market structure (the market tightness, \( \theta \) and the functional form of the matching function, \( m \)) and the instantaneous market opportunities (\( \lambda \)). Whether each selling transaction can be realized mainly depends on whether it is profitable or not for both parties.

1.3 Mathematical model

In this sub-section, we will establish the basic mathematical equations to model the interaction between investors (\( w \)) and market dealers (\( f \)) in the selling part of the security market.

We first define four key value functions since there are two types of agents (\( f \) and \( w \)) and each agent can be in two states (\( U \) means that the agent is in the idle state, \( V \) means the agent is in the occupied state). The exact meanings of the four value functions are explained below:

\( U_f \): the value of a market dealer who posts a bid price and waits for a business;

\( V_f \): the value of a market dealer who buys one share from an investor;

\( U_w \): the value of an investor who keeps one share in hand and waits for a chance to sell the share to a market dealer;

\( V_w \): the value of an investor who sells one share to a market dealer.
For any posted bid price $W$, the above four value functions satisfy the following four competitive search equations:

\[ r_{U_w} = m(\theta) (V_w - U_w) + b \]  
\[ r_{V_w} = W + \lambda(U_w - V_w) \]  
\[ r_{V_f} = P - W + \lambda(U_f - V_f) \]  
\[ r_{U_f} = \left[ \frac{m(\theta)}{\theta} \right] (V_f - U_f) - a \]

Assume the free exit and entry for any market dealer, we have the below free entry condition for the market dealer:

\[ U_f = 0 \]

The market equilibrium is characterized by a solution to the following optimal problem:

\[ (W^*, \theta^*) = \arg\max_{(W, \theta)} U_f(W, \theta) \]

s.t. (1) $U_w(W, \theta) = U_w(W^*, \theta^*)$

\[ (2) U_f(W^*, \theta^*) = 0 \]

2. Discussions

**Definition 1 (Symmetric Equilibrium):** If $(W^*, \theta^*)$ solve the optimal problem (6), then $(W^*, \theta^*)$ define a market symmetric equilibrium.

The underlying meaning of the above optimal problem is that at a predetermined market condition $(\theta^*)$, given that all other market dealers post $W^*$, any agents (either market dealers or investors) have no incentive to create an alternative market with a $W$ which is different from $W^*$. This market equilibrium definition is well consistent with the concept of Nash equilibrium.

**Proposition 1:** The market symmetric equilibrium can also be characterized by the below
optimal problem of (7), i.e. (6) and (7) are equivalent with each other and will give the same \((W^*, \theta^*)\).

\[
(W^*, \theta^*) = \text{argmax } U_w(W, \theta)
\]

\[
s.t. \quad U_f(W^*, \theta^*) = 0
\]

(7)

Linking Equation (1) - (6) together, we can solve for the four value functions \((U_f^*, V_f^*, U_w^*, V_w^*)\) and the two variables \((W^*, \theta^*)\) when assuming \((r, \lambda, P, a, b, \text{the functional form of } m)\) are all exogenous.

According to our equilibrium definition, the two values of \((W^*, \theta^*)\) exactly characterize the entire system. They can be resolved from two reduced equations summarized in Proposition 2.

**Proposition 2**: The original six-equation system (Equation (1)-(6)) can be reduced into two fundamental equations: Equation (8) is the Free entry equation and Equation (9) is the Nash equilibrium equation, which can then be used to solve for \((W^*, \theta^*)\).

**Free entry equation**:

\[
m(\theta^*)(P-W^*) - a(r+\lambda)\theta^* = 0
\]

(8)

**Nash equilibrium equation**:

\[
m'(\theta^*)(P-b) - a[r+\lambda + m(\theta^*)-\theta^* m'(\theta^*)] = 0
\]

(9)

Deriving a closed form formula for the bid-ask spread at the market equilibrium for our stylized security market under the framework of competitive search is one of the most important objectives of our paper. Once \((W^*, \theta^*)\) are solved via Equation (8) and (9), the corresponding bid-ask spread in our model will be equal to \((P-W^*)\).
Without further assumption about the functional form of the matching function \( m \), we cannot derive a closed form for the bid-ask spread from Equation (8) and (9). However, our model clearly indicates that such system parameters as \( (r, \lambda, P, a, b) \) all have a significant influence on the magnitude of the bid-ask spread at the market equilibrium. Comparative statics analysis can still be applied on those two equations via implicit function theorem (IFT) to draw many important economic implications.

In particular, if we assume a specific functional form for the matching function as \( m(\theta) = \theta^{1/2} \), which is consistent with the increasing and concave properties of \( m \), the entire system can be easily solved since Equation (9) is now independent of \( W^* \):

\[
\text{Equation (8)} \quad P - W^* = a(r+\lambda)\theta^{1/2} \tag{10}
\]

\[
\text{Equation (9)} \quad \theta^{1/2} = [(r+\lambda)^2+(P-b)/a]^{1/2}-(r+\lambda) \tag{11}
\]

Thus we have Proposition 3.

**Proposition 3:** If we assume that the matching function \( m(\theta) \) between market dealers and investors has a functional form of \( m(\theta) = \theta^{1/2} \) (so \( m \) is an increasing and concave function of \( \theta \)), the prevailing bid-ask spread at the market equilibrium can be expressed by Equation (12):

\[
P - W^* = a(r+\lambda) \sqrt{[(r+\lambda)^2+(P-b)/a]}^{1/2}-(r+\lambda) \tag{12}
\]

In a real security market, there are two closely related but different concepts about the bid-ask spread, one is the quoted bid-ask spread, the other is the effective bid-ask spread. While the quoted bid-ask spread is defined as the difference between the lowest market ask price for a security and the highest market bid price for the same security, the effective bid-ask spread is calculated as twice the difference between the actual execution price and the midquote (the
midquote is the average of the market bid and ask price) for a buy order, and twice the difference between the midquote and the actual execution price for a sell order. Here the meaning of “buy” or “sell” is considered from the viewpoint of investors. For instance, a sell order is flowed in if the current quoted market bid and ask prices are $5.00 and $5.30 respectively. (The midquote is then (5.00+5.30)/2=$5.15.) Suppose that the market deal steps in front of the previously quoted bid price and the sell order is actually executed at $5.10, the effective bid-ask spread is 2(5.15 – 5.10) = $0.10 while the quoted bid-ask spread is 5.30-5.00=0.30. Since the effective bid-ask spread better captures the cost of a round-trip order for investors by including the actual execution price in the bid-ask spread calculation, we are going to define the effective bid-ask spread below with the purpose to fit into our theoretical model which only considers the selling part of the security market. When we treat the market equilibrium bid price W* as the actual execution price and treat the price P as the midquote, the effect bid-ask spread used in our model is defined below:

**Definition 2 (Effect bid-ask spread):** The effective bid-ask spread can be expressed as twice the difference between P and W*. According to Equation (12),

\[
\text{The effective bid-ask spread}=2(P-W*)=2a(r+\lambda)\left\{[(r+\lambda)^2+(P-b)/a]^{1/2}-(r+\lambda)\right\}.
\]

(13)

3. Empirical Implications

In this section, we discuss the results of comparative statics analysis of our model, draw important empirical implications and comment the significance and contributions of our search based model.

Basing on the derived effective bid-ask spread formula Equation (13), we can clearly see
that there are five key parameters \((a, b, r, \lambda, P)\) which can significantly influence the bid-ask spread \(2(P-W^*)\). In the following, we will discuss in detail the effect of each system parameter on the bid-ask spread systemically. Although we assume a simple functional form for the matching function \(m\) when deriving a closed form bid-ask spread formula, in fact the functional form of \(m\) should have a substantial impact on the bid-ask spread, the discussion of which will deserve the work of one full paper and thus has to be omitted here. The only point to be noted is that there may exist multiple equilibriums (Lagos (2007)) if the matching function of \(m\) has the property of increasing returns with respect to its two arguments, \(v\) and \(u\), leading to two possible levels of liquidity cost (corresponding to the high bid-ask spread equilibrium and the low bid-ask spread equilibrium) for assets with almost identical cash flows (Mandal (2011)) and (Blanchard (1989)).

(1) The effect of the maintenance cost occurring to a market dealer \((a)\) on the bid-ask spread

It has long been known that the maintenance cost occurring to a market dealer is one of the most important factors which can affect the magnitude of the bid-ask spread in the security market. Traditional inventory holding cost theories (Bollena (2004)) claim that as the cost of maintaining the role of a market dealer increases, the gap between the bid price and the ask price (i.e. the bid-ask spread) will widen in order to compensate the market dealer for this unavoidable cost. According to Equation (13), the maintenance cost \(a\) has two opposite effects: the direct effect of \(a\) on the bid-ask spread is obviously positive, the indirect effect of \(a\) on the bid-ask spread is negative. Intuitively speaking, with the increase in \(a\), the total number of market dealers
(1-u+v) in the market should be reduced, either u increases or v decreases, or both, then the market tightness \( \theta = v/u \), will decrease, which may lead to the possible decrease in the bid-ask spread (Please check Equation (10)). If we assume that the positive direct effect of the maintenance cost dominates the negative indirect one, which is more likely to happen in reality, our model successfully predicts the same result as conventional inventory holding cost theories do.

**2) The effect of the dividends payoff (b) on the bid-ask spread**

The effect of the dividends payoff on the bid-ask spread is more straightforward in our model when compared with asymmetric information based models, i.e. there exists a negative relation between the dividends payoff \( b \) and the bid-ask spread. While the prediction of our competitive search based model with respect to the effect of the dividends payoff on the bid-ask spread is consistent with the conclusion of asymmetric information based theories(Howe (1992 )), both of which are supported by empirical evidence, the underlying story is totally different.

Our model can explain this negative relation without difficulty if we think of the dividends payoff as the “negative” maintenance cost to a market dealer, i.e. from the viewpoint of the market dealer, the dividends from holding one share of stock represent some positive carrying benefit. Since the main part of the market dealer’s own fund is occupied by the stock inventory holding, the more the amount of dividends paid out to the market dealer, the lower the bid-ask spread required by him.

As to the asymmetric information based theories, their underlying hypothesis is that a positive relation exists between the level of information asymmetry and the magnitude of the
bid-ask spread. Insofar as the payment of dividends signals material relevant information to the market, thus reducing information asymmetry, dividends policy may influence the bid-ask spread. Moreover, based on the above logic, an inverse relation between the dividend yield and the bid-ask spread should exist, "ceteris paribus."

Although we believe that our competitive search based explanation is more persuasive than those asymmetric information based stories, whether the negative effect of the dividends payoff on the bid-ask spread originates from our search based market friction or from the asymmetric information friction is more an empirical issue than a theoretical one, the answer of which will be left for further exploration.

(3) The effect of the stock price level (P) on the bid-ask spread

Purely looking at the formula of the bid-ask spread in Equation (13), we may draw a conclusion that there exists a monotone positive relation between the stock price level P and the bid-ask spread without hesitation. This observation is fully in agreement with Copeland and Galai’s model of information effects on the bid-ask spread (Copeland (1983)), i.e. the bid-ask spread is a positive function of the price level. The only difference is that we derive the same result from the perspective of search and matching without the assumption of asymmetric information.

One critical reason why we are greatly interested in the effect of the stock price level on the bid-ask spread is that we are attempting to apply our model to touch on a rather important and prevalent U.S. financial market phenomenon-stock splits, and its effect on stock liquidity. As we all know, stock splits are one of the intriguing anomalies in the financial market. Since they
only lead to nominal changes in stock prices and there is no any real impact on the equity ownership of shareholders, stock splits are not supposed to have any material effect on the stock price behavior and the measure of liquidity subsequent to the splits though the opposite is true in reality.

Referring to the effect of stock splits on the bid-ask spread, two strands of theories are competing with each other. The liquidity and trading range hypothesis claims that the motivation for stock splits is to bring stock prices down to a preferred trading range in order to improve liquidity. This hypothesis is strongly supported by management in practice because most managers who are in charge of stock splits do believe that the above consideration is indeed the dominating concern of their decisions on stock splits. When the bid-ask spread is utilized as our measure of liquidity, the accompanying economic implication is that the bid-ask spread should decrease after stock splits, i.e. improved liquidity follows stock splits.

Alternative asymmetric information based theory proposed by Conroy, Harris, and Benet (Conroy (1990)) suggests that stock splits with the feature or function of worsening liquidity can serve as a costly but valid signal of “favorable future prospects of the firm”. The corresponding implication is that the bid-ask spread should increase after stock splits, i.e. worsen liquidity follows stock splits.

When resorting to empirical evidence to appraise the validity of those two rival theories, the existing empirical results about the impact of stock splits on liquidity are mixed as well. The inconclusive evidence partly reflects the challenge in selecting and interpreting the proper proxy for the measure of liquidity. While the liquidity and trading range hypothesis selects the absolute
bid-ask spread (i.e. \(2(P-W^*)\)) as its measure of liquidity, the asymmetric information based theory prefers to applying the percentage bid-ask spread or relative bid-ask spread ((i.e. \(2(P-W^*)/P\)). Each theory uses the corresponding empirical results to support its own claim on the relation between stock splits and liquidity. For instance, Conroy, Harris, and Benet in the same paper find that “percentage spreads increase after splits, representing a liquidity cost to investors” for NYSE listed companies.

Summarizing the conflicting empirical results, on the one side, the bid-ask spread is positively related to the stock price level, on the other side, the percentage bid-ask spread is negatively related to the stock price. Our search based model can resolve this apparently controversial issue elegantly:

Firstly, consider Equation (13), let all the other parameters fixed, when \(P\) decreases (\(b\) also decreases in the same proportion during this process), the absolute bid-ask spread will decrease as well, showing that the absolute bid-ask spread is positively related to the stock price level \(P\);

Secondly, divide both sides of Equation (13) by \(P\) in order to obtain the formula for the percentage bid-ask spread. Roughly speaking, the numerator of the right-hand side of Equation (13) has the power of \(\frac{1}{2}\) for \(P\), the denominator has the power of \(1\) for \(P\), then the percentage bid-ask spread has the power of \(-\frac{1}{2}\) for \(P\). Thus, the percentage bid-ask spread is approximately a decreasing function of the stock price level \(P\).

(4) **The effect of the discount rate \((r)\) on the bid-ask spread**
The effect of the discount rate on the bid-ask spread is roughly positive since in most of times, the positive effect of the discount rate on the bid-ask spread dominates its negative effect. It is not difficult to understand this positive effect if we think of the discount rate as the opportunity cost of holding the stock without earning the interest for the market dealer’s own fund. The higher the discount rate is, the more the interest given up by the market dealer when he keeps an inventory of stocks, thus the higher the bid-ask spread will be required for his compensation.

(5) The effect of the instantaneous market opportunity (\(\lambda\)) on the bid-ask spread

\(\lambda\) is unique to our competitive search based model. While the effect of \(\lambda\) on the bid-ask spread is very similar to that of the discount rate \(r\) since both parameters show up in the same position in our derived bid-ask spread formula, the underlying economic meaning of \(\lambda\) is rather subtle and thus sensitive to explanation.

Formally, \(\lambda\) is defined as the market dealer’s cancellation rate for an existing deal, i.e. when the market dealer expects that the posted bid price \(W^*\) may not be appropriate for the ongoing market sentiment or economic situation, the market dealer will cancel it. From this viewpoint, we image \(\lambda\) as the measure of the instantaneous market opportunity.

Alternatively, since too low or too high \(W^*\) is equally likely to cause the market dealer to cancel the existing deal, \(\lambda\) can also be treated as the measure of the overall market uncertainty (because in our concise model, we only have one asset or stock and thus one security market). Higher \(\lambda\) means that a market dealer feels that there exists more uncertainty in the market and thus it is more likely for the posted bid price \(W^*\) unfit for the current market condition, leading
to a deal cancelled more often.

In sum, we conclude that:

**Proposition 4 (Determinants of the bid-ask spread):**

1. The bid-ask spread is positively related to the market dealer’s maintenance cost $a$, the discount rate $r$ and the instantaneous market opportunity $\lambda$;
2. The bid-ask spread is negatively related to the dividends from holding one share $b$.
3. The bid-ask spread is positively related to the stock price level $P$; but the percentage bid-ask spread is negatively related to the stock price level $P$.

Moreover, there are several important caveats which need to be further discussed and clarified below.

**(A) De-centralized market vs. centralized market**

 Basically speaking, if there is a centralized market for the transaction of an asset, most of the market search friction considered here will become trivial. Thus, from this narrowest viewpoint, our model applies to over the counter market, the foreign exchange market, and the other de-centralized markets such as the real estate market.

However, from the perspective of the normal market operation, the market dealer (or any other agent with the same responsibility but the different title in an asset market) still needs to be credited with providing the necessary matching service to both actual buyers and sellers. Just because of the existence of this type of agent, the market search friction in a centralized market can be reduced to the current minimum level. Thus, our model can also be utilized to assess the magnitude of liquidity cost in terms of the bid-ask spread in a centralized market.
Furthermore, in history, centralized markets all evolve from the initial de-centralized markets. Our search based model can earn its merits from this viewpoint. When considering the compensation for a role of a market dealer playing in a centralized market as the form of the bid-ask spread, we need to compare two states, one is the real centralized market, and the other is the imaginary de-centralized market, even if the de-centralized market is not the current status of the market. To be more specific, only if we explore how difficult it is to meet the other side of the transaction in the imaginary de-centralized market, we can then evaluate accurately how well the current centralized market provides to investors and justify the amount of the bid-ask spread required by a market dealer in the real centralized market. Hence, the market search and matching friction modeled in this manuscript, should be embedded implicitly in the bid-ask spread for any centralized market.

More importantly, our search based model is distinguished from traditional search models by how the search process works. While traditional search models assume 1) the random matching between the types of agents such as workers and firms, 2) the determination of W* by bargaining once they meet, our search model lets market dealers to post a widely known bid price W* in public ex ante in order to direct or attract the arrival of investors. Due to the above feature, our model can be classified as “directed search and posting” or “competitive search”(Moen (1997)) and (Shimer (1996)). Furthermore, our symmetric equilibrium lets all market dealers play the same strategy at equilibrium, i.e. post the same bid price W* in the entire market. In this way, our search model well mimics the operation of NASDAQ stock market.

(B) Search and matching friction cost vs. Inventory-holding cost and asymmetric
information cost

The earlier version of theories of the bid-ask spread focused so much on the concept of inventory-holding cost for a market dealer that it ignored the information content represented by the bid-ask spread. Although modern theories of the bid-ask spread (Bollena (2004)) pay enough attention to asymmetric information cost associated with two types of investors in the market, the random or noise investors and the informed investors, to our knowledge, there is no such a theory or model except ours that puts the important search and matching friction cost into consideration. The relative weight of the inventory-holding cost and the search and matching cost is basically an empirical issue, on which our model has the potential to shed light. Basing on our bid-ask spread formula in Equation (13), though the two terms are coupled with each other in our model, if the search and matching cost is roughly estimated as by W* and the inventory-holding cost is proxied by the maintenance cost a. So the ratio of the two types of costs will be W*/a.

(C) The number of market dealers: Competition vs. monopoly

Another interesting result is related to the number of market dealers which can be supported by the market. Since the market tightness θ* = v/u, if u is known, we can pin down the total number of market dealers at the market equilibrium, which is equal to (1-u +v), here, 1-u is the number of market dealers who have a business, v is the number of market dealers who are idle.

One salient feature of our model is that we don’t assume the total number of market dealers be fixed, ex ante. This number is endogenously decided by the system via a free entry and exit condition for market dealers, which is the reason why our model is called the “competitive”
search model. Intuitively, some researchers give a pre-emptive monopoly position to a market dealer. Therefore their models may lead to a comparatively higher bid-ask spread owing to the monopoly profit.

4. Calibration and simulation

In this section, we calibrate the key parameters of our bid-ask spread formula according to the typical values of the stock market in order to show the impacts of those parameters on the bid-ask spread quantitatively. Our simulation results illustrate that the bid-ask spread indicated by our competitive search based model well fits into empirical observations. However, it should be noted that the choices of values of model parameters may have a significant effect on the magnitude of the bid-ask spread.

Table 1 summarizes the parameter values which will be used in our model.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Notation</th>
<th>Typical Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maintenance cost occurring to a market dealer</td>
<td>a</td>
<td>$15 (1.5P)</td>
</tr>
<tr>
<td>Dividends produced each period by one share of stock</td>
<td>b</td>
<td>$9.70 (0.97P)</td>
</tr>
<tr>
<td>Stock price level</td>
<td>P</td>
<td>$10</td>
</tr>
<tr>
<td>Discount rate</td>
<td>r</td>
<td>0.04</td>
</tr>
<tr>
<td>Instantaneous market opportunity</td>
<td>( \lambda )</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Table 1 Parameter values
Figure 1 shows the positive effect of the maintenance cost \((a/P)\) on the relative bid-ask spread \((2(P-W^*)/P)\) when the values of the other parameters from Table 1 are fixed. We let the maintenance cost change from one time to five times of the stock price level \(P\).

![Figure 1](image)

**Figure 1**

Figure 2 shows the negative effect of the dividend yield \((b/P)\) on the relative bid-ask spread \((2(P-W^*)/P)\) when the typical values of the other parameters are still used from Table 1. The possible values of the dividend rate range from 0.95 \(P\) to 0.99\(P\). Here \(P\) is the stock price level.
Figure 2

Figure 3 shows the positive effect of the discount rate \( r \) on the relative bid-ask spread \( \frac{2(P-W^*)}{P} \) when the typical values of the other parameters are used from Table 1. The possible values of the discount rate range from 0.02 to 0.06.

Figure 3
Figure 4 shows the positive effect of the market instantaneous opportunity ($\lambda$) on the bid-ask spread ($2(P-W^*)/P$) when the typical values of the other parameters are used from Table 1. The possible values of the market instantaneous opportunity range from 0.03 to 0.07.

![Effect of instantaneous market opportunity on bid-ask spread](image)

**Figure 4**

Figure 5 (a) and (b) show the effects of the stock price level ($P$) on both the relative (percentage bid-ask spread) (Figure 5(a)) and the absolute (dollar bid-ask spread) (Figure 5(b)) when the typical values of the other parameters are used from Table 1 except that we keep the dividends payoff $b$ as a constant ratio of the stock price level $P$, i.e. $b=0.97P$. The possible values of the stock price level range from $5$ to $15$. We can see clearly that the absolute bid-ask spread is positively related to the stock price level; while the percentage bid-ask spread is negatively related to the stock price level.
In addition, we can also evaluate the relative weight of the inventory-holding cost and the search and matching cost by $W^*/a$, showed in Figure 6 where the search and matching cost
becomes less important when the maintenance cost increases.

![Graph showing the relative weight of search and matching cost and inventory-holding cost against maintenance cost (a/P).](image)

**Figure 6**

**5. Conclusion and future directions**

In this paper, the optimal behaviors of both market dealers and investors are simultaneously investigated under the framework of competitive search theory. Four useful value functions for both agents are established to represent the corresponding utilities obtained when staying in two distinct states, the idle state and the occupied state.

Our competitive search based model shows that the bid-ask spread charged by market dealers is legitimated by their liquidity service provided to actual buyers and sellers in the security market. The more difficult for an investor to locate a counter party to trade in a de-centralized market, the higher the bid-ask spread required by market dealers in a centralized market.
Moreover, our model stress that the ultimate bid-ask spread prevalent in the centralized market will also be cut down by the competition among market dealers. Our derived bid-ask spread formula indicates that the magnitude of the bid-ask spread is affected by many factors such as the cost for maintaining a market dealer’s position, the (risk-free) discount rate, the overall market uncertainty, the dividends payoff and the stock price level, etc.

Our model’s theoretical prediction on the effect of dividends policy on the bid-ask spread is consistent with asymmetric information based theories and empirical observations. Our model successfully addresses the apparent controversial issue on the effect of the stock price level on the absolute bid-ask spread and on the percentage bid-ask spread without the assumption of asymmetric information.

Several directions for the future work are suggested, including introducing more detailed economic analysis of the influencing parameter $\lambda$; considering the more complicated case of an integrated two-side search covering the price formation mechanism for both bid price and ask price; and combining the theoretical predictions with the empirical results to further testify the validity of the proposed model.
Appendix A  Notation Table

\( f: \) market dealers.

\( w: \) investors.

\( P: \) the instantaneous price at which a market dealer can dispose of one share of stock. (Since we only model the selling part of the market, \( P \) can be treated as parameter.)

\( W: \) the instantaneous bid price posted by a market dealer.

\( W^*: \) the instantaneous bid price at the market equilibrium.

\( 2(P-W^*): \) twice the difference between \( P \) and \( W \), denoting the effective bid-ask spread at the market equilibrium.

\( 1: \) the initial number of investors, normalized to 1.

\( u: \) the number of investors who don’t sell their shares.

\( 1-u: \) the number of investors who sell their shares, which is also the number of market dealers who have business.

\( v: \) the number of market dealers who are idle.

\( 1-u+v: \) the total number of market dealers including occupied and idle.

\( a: \) the maintenance cost occurring to a market dealer when posting a bid price in the market.

\( b: \) the dividends produced each period by one share as one Lucas tree.

\( \theta: \) the market tightness, equals \( \frac{v}{u} \).

\( \theta^*: \) the market tightness at the market equilibrium.
**m (θ):** the matching technology function between f and w, which is an increasing and concave function of θ. Thus, the meeting rate for investors is m (θ), for market dealers is m (θ)/ θ, respectively.

λ: the instantaneous market opportunity or the overall market uncertainty.

r: the (risk-free) discount rate.

**State ”U”**: means that it is in the idle state.

**State ”V”**: means that it is in the occupied state.

U_f : the value of a market dealer who posts a bid price and waits for a business.

V_f : the value of a market dealer who buys one share from an investor.

U_w : the value of an investor who keeps one share in hand and waits for an chance to sell.

V_w : the value of an investor who sells one share to a market dealer.
Appendix B  Proofs of Propositions

Proposition 1: Market equilibrium (Equivalence of two optimal problems: (6) and (7))

From the optimal problem (6),

\[ L_1 = U_f(W, \theta) + \mu [U_w(W, \theta) - U_w(W^*, \theta^*)] \]

F.O.C. for \( W \):

\[
\frac{\partial L_1}{\partial W} = \frac{\partial U_f}{\partial W} + \mu \frac{\partial U_w}{\partial W} = 0
\]

for \( \theta \):

\[
\frac{\partial L_1}{\partial \theta} = \frac{\partial U_f}{\partial \theta} + \mu \frac{\partial U_w}{\partial \theta} = 0
\]

From the optimal problem (7),

\[ L_2 = U_w(W, \theta) + \eta U_f(W, \theta) \]

F.O.C. for \( W \):

\[
\frac{\partial L_2}{\partial W} = \frac{\partial U_w}{\partial W} + \eta \frac{\partial U_f}{\partial W} = 0
\]

for \( \theta \):

\[
\frac{\partial L_2}{\partial \theta} = \frac{\partial U_w}{\partial \theta} + \eta \frac{\partial U_f}{\partial \theta} = 0
\]

When set \( \mu = 1/\eta \), those two sets of conditions are equivalent with each other.

Proposition 2: Two equilibrium equations

(1) Derive the free entry equation:

Step one: Solve for \( U_f \).

For a market dealer, (3)-(4), we get:

\[
r(V_f - U_f) = P - W + a - (V_f - U_f)(\lambda + m/\theta)
\]

\[
(V_f - U_f) = (P - W + a) / (r + \lambda + m/\theta)
\]

Put the above relation back into (4),

\[
rU_f = -a + (m/\theta) * (V_f - U_f) = -a + (m/\theta) * [(P - W + a) / (r + \lambda + m/\theta)]
\]
\[-a + \frac{m(P-W+a)}{\theta(r+\lambda)+m}\]

\[= \frac{m(P-W)-a\theta(r+\lambda)}{\theta(r+\lambda)+m}\]

So \(U_f = \frac{m(P-W)-a\theta(r+\lambda)}{r\theta(r+\lambda)+rm}\)

Step two: Use (5), since \(U_f = 0\), the numerator has to be zero,

i.e. \(m(P-W)-a\theta(r+\lambda) = 0\)

At market equilibrium, \(W^*\) and \(\theta^*\), we get:

\[m(\theta^*)(P-W^*) - a(r+\lambda)\theta^* = 0\]

(2) Derive the Nash equilibrium equation:

Step one: Solve for \(U_f\).

According to the result derived from the free entry equation, we have known that: \(U_f = \frac{m(P-W)-a\theta(r+\lambda)}{r\theta(r+\lambda)+rm}\)

Step two: Solve for \(U_w\).

For an investor, (2)-(1), we get:

\[r(V_w-U_w) = W-b-(V_w-U_w)(\lambda+m)\]

\[(V_w-U_w) = \frac{W-b}{r+\lambda+m}\]

Put the above relation back into (1),

\[rU_w = b+m(V_w-U_w) = b+m\left(\frac{(W-b)}{(r+\lambda+m)}\right)\]

\[= \frac{mW+(r+\lambda)b}{r+\lambda+m}\]

So \(U_w = \frac{mW+(r+\lambda)b}{r(r+\lambda+m)}\)

Step three: Solve the optimal problem (6)

Use the LaGrange method, set up two Lagrange multipliers \(\mu\) and \(\eta\) since there are two
constraints (actually the second constraint has no effect on the first order conditions).

\[ L = U_r(W, \theta) - \mu [U_w(W, \theta) - U_w(W^*, \theta^*)] - \eta[U_f(W^*, \theta^*)] \]

Put \( U_r(W, \theta) \) and \( U_w(W, \theta) \) derived from Step one and Step two into \( L \), we get:

\[ L = \frac{[m(P-W)-a(\theta+\lambda)]}{r\theta(r+\lambda)+rm} \]

- \( \mu[rU_w^*(r+\lambda+m)-mW-(r+\lambda)b] \)
- \( \eta[U_f(W^*, \theta^*)] \)

Here, \( U_w^* \) is the abbreviated form of \( U_w(W^*, \theta^*) \).

Now consider:

First order condition for \( W \):

\[ -m/[r\theta(r+\lambda)+rm] + \mu m = 0 \]

So, \( \mu = 1/[r\theta(r+\lambda)+m] = 1/[\theta(r+\lambda)+m] \}

First order condition for \( \theta \):

(Note that \( m \) is also a function of \( \theta \))

\[ (1/r) \{[m'(P-W)-a(r+\lambda)]/[\theta(r+\lambda)+m] - [m(P-W)-a(\theta+\lambda)]/[\theta(r+\lambda)+m] \}^2 - \mu (rU_w^*m'-m'W) = 0 \]

Then put \( \mu = 1/[\theta(r+\lambda)+m] \} \) into the above equation, delete the factor of \( 1/r \) and \( 1/[\theta(r+\lambda)+m] \} \) on both items,

\[ [m'(P-W)-a(r+\lambda)] - [m(P-W)-a(\theta+\lambda)]/[\theta(r+\lambda)+m] - (rU_w^*m'-m'W) = 0 \]

Apply the free entry condition, at market equilibrium \((W^*, \theta^*)\), we know that: \( m(\theta^*)(P-W^*)-a \theta^*(r+\lambda) = 0 \), So the middle item is equal to zero.

Finally only the first and last items are left.
\[(m'(P-W)-a(r+\lambda)- (rU_w*m'-mW)=0\]
\[m'(P-W)-a(r+\lambda)- m'(rU_w*-W)=0\]
\[m'(P- rU_w*)-a(r+ \lambda)=0\]

Moreover, replace \(rU_w*\) by \(r U_w(W*, \theta*)=[mW*+(r+\lambda)b]/ (r+ \lambda+m)\),
\[m'[P(r+\lambda+m)- mW*-(r+\lambda)b] -a(r+\lambda) (r+\lambda+m)=0\]
\[m'[P(r+\lambda)+mP- mW*-(r+\lambda)b] -a(r+\lambda) (r+\lambda+m)=0\]

According to the free entry condition:
\[m(\theta^*)(P-W*)-a \theta^*(r+\lambda)=0\]
so \(mP-mW*= a \theta^*(r+\lambda)\),

Thus, at market equilibrium, the F.O.C. for \(\theta\) changes into:
\[m'[P(r+\lambda)+ a \theta^*(r+\lambda)-(r+\lambda)b] -a(r+\lambda) (r+\lambda+m)=0\]
\[m'(P+ a \theta*-b) -a(r+\lambda+m)=0\]
i.e. \(m'(\theta^*)(P-b) -a[r+\lambda+ m(\theta^*)-\theta^* m'(\theta^*)]=0\)

**Proposition 3: Bid-ask spread formula**

This proposition has already been proved in the paper. The brief proof is reproduced here for your reference.

If \(m(\theta)=\theta^{1/2}\),

The free entry equation changes into: \(\theta^{1/2}(P-W*)-a(r+\lambda) \theta^*=0\)
i.e. \(P-W*=a(r+\lambda) \theta^{1/2}\)

The Nash equilibrium equation changes into:
\[0.5 \theta^{*1/2}(P-b)-a(r+ \lambda+ \theta^{*1/2}-0.5 \theta^* \theta^{*-1/2}) =0\]
0.5 \( \theta^{* -1/2} (P-b) - a(r+\lambda + 0.5\theta^{*1/2}) = 0 \)

Let \( \theta^{*1/2} = X \), then, \( \theta^{*-1/2} = 1/X \), the above equation can be changed into:

\[ X^2 + 2(r+\lambda)X - (P-b)/a = 0 \]

Solve the above quadratic equation, get X,

\[ \theta^{*1/2} = X = [(r+\lambda)^2 + (P-b)/a]^{1/2} - (r+\lambda) \]

Thus, the bid-ask spread is:

\[ P-W^* = a(r+\lambda) \theta^{*1/2} = a(r+\lambda) \{[(r+\lambda)^2 + (P-b)/a]^{1/2} - (r+\lambda)\} \]

**Proposition 4: Determinants of bid-ask spread**

It is easy to check out the signs of the first derivations of our bid-ask spread with respective to those determinants.
References


